Compound Interest

Compound Interest – Future Value Formula

Derivation of the Compound Amount Formula

Conversion Period	Computation of interest	New Principal
1	$S = P + iP$	$S = P(1+i)$
2	$S = P(1+i) + iP(1+i) = P(1+i)(1+i)$	$S = P(1+i)^2$
3	$S = P(1+i)^2 + iP(1+i)^2 = P(1+i)^2(1+i)$	$S = P(1+i)^3$
	…	…
n	$S = P(1+i)^{n-1} + iP(1+i)^{n-1} = P(1+i)^{n-1}(1+i)$	$S = P(1+i)^n$

This indutive process implies the principle of compound interest to a given principal *P.*
Steps in this derivation → *repeated applications of the distributive property and the laws of exponents.*

Compound Amount Formula: $\boldsymbol{S = P(1+i)^n}$

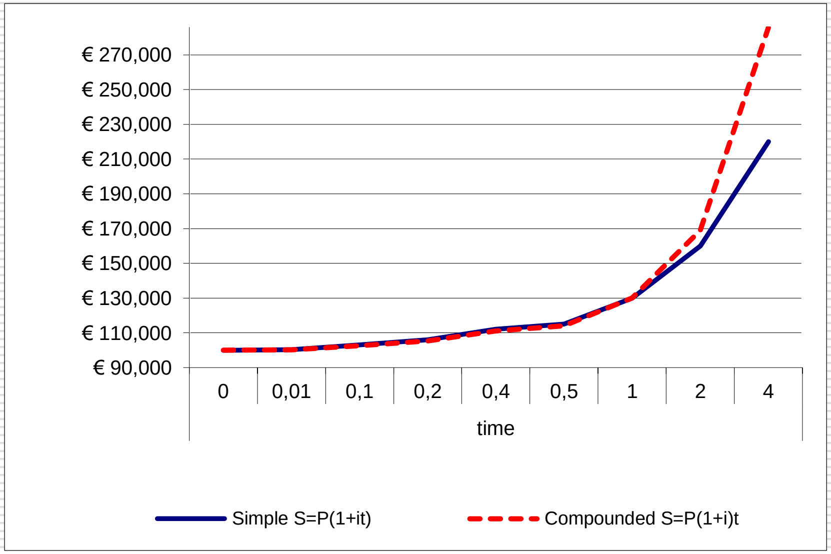
Accumulation Factor (or function)

(Basic tool to move money on the time line using compound interest)

Compound Interest – Future Value Formula

- The calculation of simple interest is always based on the original principal, but compound interest is added to the principal in such a way that each time interest is computed, it is based on a larger principal than the previous time.

Growth due to Interest



n(in years)	Simple	Compounded
0	100,000 €	100,000 €
0,01	100,300 €	100,263 €
0,1	103,000 €	102,658 €
0,2	106,000 €	105,387 €
0,4	112,000 €	111,065 €
0,5	115,000 €	114,018 €
1	130,000 €	130,000 €
2	160,000 €	169,000 €
4	220,000 €	285,610 €

n<1	***Money invested at simple interest grows faster than money invested at compounded interest***
n=1	***The future value for the simple and compound interest is the same.***
n>1	***Money invested at compound interest grows faster than money invested at simple interest.***

The Present Value Formula and Discounting

Institutions and investors frequently set financial goals or have debt obligations that require current investments in order to provide the capital to meet those goals at some future date.

Since the time and value of the future monies are tipically known:

The current investments will be the present value computed at the best available compound rate

(definition) The Present Value is the sum of money that will increase at compound interest to some known future amount at a given data in the future.

Compound Interest: The amount of interest earned during the term of the investment

Discount: The difference between the present value and the future amount (same definition as in simple interest rate).

The Present Value Formula and Discounting

<u>Compound Amount Formula to the Present Value:</u>

Steps

1) The Future Value formula $S = P(1+i)^n$

2) Transpose so P is on the left side $P = \dfrac{S}{(1+i)^n}$

Present Value Formula: $P = S(1+i)^{-n} = Sv^n$

Discount Factor
or
the Present Worth of 1 Factor
(reciprocal of the acummulation factor)

Theory of Interest
Compound Discount

Compound discount certainly completes the comparison between *Simple interest* vs *Compound Interest* and *Simple Discount* vs *Compounded Discount*.

Compound Interest

- Dominant tool in modern financial transactions.

Simple Discount

- Used frequently in establishing yield rates for the sale of short-term financial instruments (e.g. treasure bills).
- Used in conjunction with compound interest for fractions of an interest period.

Simple Interest

- Still in limited use for short-term notes.
- Used in conjunction with compound interest for fractions of an interest period.

Theory of Interest
Compound Discount

i(m) Nominal Rate is the annual percentage rate.

Conversion Period (or **interest period** or **capitalization period**) is the time between successive computations of interest;

m is the number of conversion periods per year;

i interest rate per conversion period and is equal to the nominal rate i(m) divided by number of periods per year. **i = i(m)/m;**

Nominal Rates and Effective Interest

- ***<u>Annual Effective Rate (Annual Percentage Yield – APY):</u>***

(definition) Based on the fact that equivalent interest rates produce the same return for a given investment, annual effective rate is an annually converted rate $i(1)$ that gives the same interest earnings as the nominal rate $i(m)$ converted m times per year, where $m \neq 1$.

Variation of the definition of the effective rate of interest

Derivation of the Formula

$S = (1+i)^1$ for €1 invested — compound interest at rate $i(1)$ for one year

$S = \left[1+\frac{i(m)}{m}\right]^m$ for €1 invested — nominal compound rate $\frac{i(m)}{m}$ for one year

$(1+i) = \left[1+\frac{i(m)}{m}\right]^m$ — Set them equal to each other.

$\boxed{i = \left[1+\frac{i(m)}{m}\right]^m - 1}$ — Transpose the 1 to the right side.

Annual Effective Rate Formula (APY)

Question: when the annual efective rate is equal to the nominal rate ?

Nominal Rates and Effective Interest

- ***Nominal Rate formula:***

Solving the Annual Effective Rate Formula for the nominal rate *i*(*m*)

$$(1+i)=\left[1+\frac{i(m)}{m}\right]^{m}$$

$$(1+i)^{1/m}=\left[1+\frac{i(m)}{m}\right]^{m\frac{1}{m}}$$

Take the 1/*m* power of each side.

$$(1+i)^{1/m}=1+\frac{i(m)}{m}$$

Eliminate the power on the right.

$$\frac{i(m)}{m}=(1+i)^{\frac{1}{m}}-1$$

Transpose terms to solve.

$$\boxed{i(m)=m\left[(1+i)^{\frac{1}{m}}-1\right]}$$

Solve for the ***nominal rate***.

- ***Annual Effective Rate of Compound Discount:***

A nominal rate of discount *d(m)* payable *m* times per year → the rate is *d(m)/m* for each conversion period.

Compound Rate

Compound rate: is a growth rate that equates the present value and the future value for a number of compounding periods.

$$S = P(1+i)^n$$

Compound Amount formula

$$\frac{S}{P} = (1+i)^n$$

Divide both sides by *P*.

Compounded Annual Growth Rate is a quantitative finance specific term for the **geometric mean** that provides a constant rate of return over the time period of the investment.

$$\boxed{i = \left(\frac{S}{P}\right)^{\frac{1}{n}} - 1}$$

Transpose the 1 and write in formula order.

Compound Rate Formula

Finding the Time for an investment to Grow

- Investors, accountants, and finantial managers occasionally need to find how long it will take a sum of money to increasea certain percent or even double.

Finding the time for an investment to reach a known future value is only slightly more complicated than simply finding the rate.
(***e.g.*** If a fund, growing at a cerytain rate, will take too long to reach the desired future value, the manager may have to look for higher risk with a larger rate of increase.)

Derivation of the Formula
(Time = Number of Conversion Periods - *n*)

$S = P(1+i)^n$	Compound Amount formula.
$\frac{S}{P} = (1+i)^n$	Divide both sides by *P*.
$\ln\left(\frac{S}{P}\right) = \ln(1+i)^n$	Take the natural log of each side.
$\ln\left(\frac{S}{P}\right) = n[\ln(1+i)]$	Use the property log $M^N = N \log M$.
$\boxed{\frac{\ln\left(\frac{S}{P}\right)}{\ln(1+i)} = n}$	Divide both sides by *ln(1+i).*

Formula for the Number of Conversion Periods

Finding the Time for an investment to Grow

- The determination of the time for an investment to grow must consider three cases. When we compute the value of *n it will be almost always be a decimal fraction requiring interpretation.*

Interpretations when finding the time:

1) To *the nearest number* of periods will require rounding to a whole number by standard rounding techniques. If the decimal is 0.5 or larger we round up; otherwise, drop the fraction.

2) At *least* enough periods to reach the amount of the desired future value will require that we always round up. When the decimal fraction is very small(***e.g***. 0.01), we can round down.

3) To *the day* will require that simple interest be given for part of a conversion period. We compute the future value based on the whole number part of the computed n, but we use simple interest to take this future value to the desired goal.

Equations of Value to Find the Unknown

- **(definition)** An equation of value is a mathematical expression that equates the value of several pieces of money at some chosen date called the ***focal date***.

Determines whether certain pieces of money are moved forward to a future value or moved backward to a present value

A well- planned time line diagram is useful to write the equation o f value

Example:

A businessman signs a note for €8500 due in 18 months at 7.5%(4). With some extra income, he makes a €4200 payment after 1 year. How much will he owe on the original due date?

- The original due date is chosen as the focal date and move the payment and amount to that day. Settlement cost: difference between the payment and the amount. Money will be moved to the focal date using compound interest

Time Line Diagram

Present	**1 year(4 quarters)**	**18 months(6 quarters)**
€8500	Payment 1	x = payment 2
	$n = 2$	S_1
$n = 6$		S_2

Equation

$$X = S_2 - S_1$$
$$= 8500(1{,}01875)^6 - 4200(1{,}01875)^2$$
$$= €9502{,}21 - €4358{,}98 = €5143{,}23$$

Continuous Compounding

- ***Continuous Compounded Annual Effective Interest Rate:***

When we compute effective interest rates (APY) we observe that increasing the number of compounding periods per year also increased the effective rate but the increase slows down.

If *m* become infinitly large → *interest compounds more often than every second.*

Compounded continuously

Derivation of the Formula

Evaluate the limit in the expression for the effective rate: $i = \lim_{m\to\infty}\left(1+\frac{1(m)}{m}\right)^m - 1$

Knowing the definition of natural log base *e*: $\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m$ *m* is a positive real number.

Using the theorem of limits : $\lim_{x\to\infty}(g(x))^j = (\lim_{x\to\infty} g(x))^j$

From the compound rate formula let $k = \frac{m}{i(m)} = \frac{m}{\delta}$, so that $m = k\delta$ .As $m \to \infty$ we see $k \to \infty$.

Substituting : $\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{k\delta} = \left[\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k\right]^\delta = e^\delta$ (limit goes to *e*)

Continuous Compounding Annual Effective Rate: $i = e^\delta - 1$

Continuous Compounding

Continuously Compounded Nominal Rate :

Looking this from the opposite perspective

Assumption: ***Desire for a particular effective rate of interest for an investment.***

Requirements
- know the nominal rate compounded continuously that must be earned.
- find nominal rates at various compounding intervals.

Derivation of the Formula

Use the properties of logs on the ***effective rate formula*** for continuous compounding

$\ln(M^N) = N\ln(M)$ and $\ln(e) = 1 (\log e = 1)$

Solve for δ algebraically:

$1 + i = e^{\delta}$	Transpose the 1.
$\ln(1+i) = \ln(e^{\delta})$	Take the ln of each side.
$\ln(1+i) = \delta \ln e$	Use property of logs.
$\ln(1+i) = \delta$	ln(e)=1

Nominal rate of interest compounded continuously : $\delta = \ln(1+i)$
(*Force of Interest* function for continuously compounded interest)

Annual Effective Rate

Continuous Compounding

- ***Future and Present Value Formulas for Continuous Compounding :***

$S = P(1+i)^n$ → amount (or *future value*) formula in which i equals the rate per period and n represents the total interest periods.

$$S = P(1+i)^n$$ Future value formula

$$S = P(1+e^{\delta}-1)^t$$ Replace i by the effective rate and n by the number of years.

$$S = P(e^{\delta})^t$$ Simplify inside the parentheses.

$$\boxed{S = Pe^{\delta t}}$$ Eliminate the parentheses for the final form of the formula.
(time is expressed in years)

Future Value Formula

Present Value Formula: $S = Pe^{\delta t} \rightarrow Se^{-\delta t} = Pe^{\delta t}e^{-\delta t} \rightarrow \boxed{P = Se^{-\delta t}}$